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The Neumann series as a fundamental solution of the two-dimensional convection–diffusion equation with variable velocity

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Abstract The fundamental solution of the two-dimensional convection–diffusion equation with variable coefficients and its adjoint equation are obtained in complex form in terms of the unknown density of two equivalent uniquely solvable Volterra integral equations of the second kind whose analytical solutions are given explicitly as convergent Neumann series. The Volterra integral equations are obtained by integrating the complex form of the original differential equations, without additional change of variables as proposed by previously authors. In the numerical examples, cases corresponding to non-self-adjoint operators are considered. As a validation, the proposed approach is used to derive the fundamental solution of the adjoint to the convection–diffusion equation with constant velocity. In this case, the series solution can be evaluated analytically. For more general velocity fields, the recursive terms of the series can be evaluated by symbolic computation or numerical integration.

Keywords Fundamental solution · Two-dimensional convection–diffusion equation · Variable velocity

1 Introduction

Many engineering problems are modelled by the convection–diffusion equation. Numerical solutions have been successfully obtained using a variety of approaches. The boundary element method (BEM) is a very attractive solution technique, since it reduces the dimension of the problem by one unit. The basis of the BEM method is to transform the original partial differential equation (PDE) that describes a given physical problem into an equivalent integral equation, either by means of the Green's representation formula (direct method), or by means of a continuous distribution of singular solutions of the PDE over the boundaries (indirect method). The unknowns in the integral formulation of the boundary-value problem are either the primitive variables on the boundary (direct formulation) or fictitious surface densities of the singular solutions (indirect formulation). Since the solution of

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the integral equation satisfies the governing field equation exactly, one seeks to satisfy the imposed boundary conditions.

The integral-representation formula for the convection–diffusion equation with variable velocity field is obtained from the corresponding Green's second identity using the fundamental solution of the adjoint PDE (for more details see [1]). The formulation of a direct BEM numerical method is based on this integral representation. With the indirect BEM method and the method of fundamental solution (MFS), continuous and discrete distributions of the fundamental solution of the original differential equation are used. These three numerical approaches are well suited for boundary-value problems governed by the convection–diffusion equation with variable coefficients. However, an expression for such fundamental solutions is required.

A fundamental solution (Green's function) is a particular solution of a partial differential equation with the delta function as an inhomogeneous term; consequently, the Green's function is singular at the pole of the delta function. As a particular solution, the fundamental solution is not uniquely defined since we can always add any solution of the homogeneous equation. Although it is theoretically possible to guarantee the existence of the fundamental solution of the convection–diffusion equation with variable velocity and its adjoint equation, closed-form expressions of such fundamental solutions are known only for simple cases, corresponding to equations with constant coefficients, or coefficients with a specific variation in space that allows analytical integration (see [2,3]).

By use of a complex conjugate set of variables, it is possible to transform a linear elliptic two-dimensional PDE into a hyperbolic PDE. Using this transformation, Vekua [4] and Garabedian [5] derived the Riemann functions of elliptic equations and their adjoint equations as the unknown density of uniquely solvable Volterra integral equations of the second kind. In contrast with the fundamental solutions, the Riemann functions are regular solutions of homogeneous equations satisfying given initial conditions. Vekua [4] and Garabedian [5] also demonstrate that the fundamental solutions of the same equations can be found in terms of a given combination of those Riemann functions, the fundamental solution of Laplace's equation, and other regular functions.

By introducing a change of variable in the transformed complex plane, Bergman [6] obtained an infinite-series solution of the fundamental solution of the adjoint of a general elliptic equation. Although Bergman's series gives an explicit form of the fundamental solution, its numerical evaluation is tedious (for a more detailed discussion of these difficulties see [7]). To avoid the difficulties with the evaluation of Bergman's series, Hill and Porter [7] (see also [8,9]) proposed solving a Volterra integral equation of the second kind for Bergman's new function as the unknown density. Hill and Porter's Volterra integral equation is obtained by integrating the PDE in the complex plane obtained by Bergman for the new function, using an approach similar to that used by Vekua [4] and Garabedian [5] to find the Riemann functions.

In this work, we propose a new way of obtaining the fundamental solutions of the convection–diffusion equation with variable velocity and its adjoint equation, in terms of the unknown density of two equivalent uniquely solvable Volterra integral equations of the second kind whose analytical solutions are given by convergent Neumann series. The resulting Neumann series give explicit expressions for these functions, as in the case of Bergman's series, but without the difficulties encountered in the evaluation of the recursive terms. Moreover, the integral equations are obtained directly from the complex form of the original differential equations, as is the case in the Vekua and Garabedian's approach for the Riemann functions, circumventing an additional change of variables used by Bergman [6] and Hill and Porter [7].

As a test of the proposed scheme, the fundamental solution of the adjoint of the convection–diffusion equation with constant velocity is considered. In this case, the recursive integrals in the Neumann series can be found analytically. A comparison of the obtained solution with the known closed-form fundamental solution is also made. Due to the non-uniqueness of the fundamental solution, solutions of the homogeneous equation are introduced to obtain different forms of the fundamental solution. For more complex variations of the velocity field, the recursive terms of the series can be evaluated by symbolic computation or numerical integration. Alternatively, the obtained Volterra integral equation can be solved numerically using a scheme similar to that proposed by Hill and Porter [7].

2 Integral-representation formulas, Riemann and fundamental solutions

The two-dimensional steady-state convection–diffusion equation for a homogeneous and isotropic medium, including first-order reaction, can be written in the form

$$\mathcal{L}\left[u(\vec{x})\right] = \nabla^2 u(\vec{x}) + a(\vec{x})\frac{\partial u(\vec{x})}{\partial x} + b(\vec{x})\frac{\partial u(\vec{x})}{\partial y} + d(\vec{x})u(\vec{x}) = f(\vec{x}), \tag{1}$$

where $a(x, y) = -v_x(x, y)/D, b(x, y) = -v_y(x, y)/D, d(x, y) = k(x, y)/D$ and $f(\vec{x}) = g(\vec{x})/D$, v_x and v_y are the components of the convection velocity vector $\vec{v}$ in the x and y direction, respectively, k is a reaction coefficient, g is a production term, and D is the diffusion coefficient.

Solutions of Eq. (1) can be given in terms of a surface integral-representation formula (see [1]);

$$\begin{aligned} c(\vec{x})u(\vec{x}) = &\int_\Gamma \frac{\partial u(\vec{x}_o)}{\partial \vec{n}_{\vec{x}_o}}\, \Phi(\vec{x}, \vec{x}_o)\, \mathrm{d}\Gamma_{\vec{x}_o} - \int_\Gamma \frac{\partial \Phi(\vec{x}, \vec{x}_o)}{\partial \vec{n}_{\vec{x}_o}} u(\vec{x}_o)\mathrm{d}\Gamma_{\vec{x}_o} \\ &+ \int_\Gamma \left[a(\vec{x}_o)n_1(\vec{x}_o) + b(\vec{x}_o)n_2(\vec{x}_o)\right] u(\vec{x}_o)\Phi(\vec{x}, \vec{x}_o)\mathrm{d}\Gamma_{\vec{x}_o} \\ &+ \int_\Omega f(\vec{x}_o)\, \Phi(\vec{x}, \vec{x}_o)\, \mathrm{d}\Gamma_{\vec{x}_o}, \end{aligned} \tag{2}$$

where $\overrightarrow{n}$ is the outward unit normal vector, $c(\vec{x}) = \frac{\theta}{2\pi}$, and θ is the solid internal angle of the boundary at the point $\vec{x} \in \Gamma$; $c(\vec{x}) = 1$ when $\vec{x} \in \Omega$ and $c(\vec{x}) = 0$ when $\vec{x} \notin \overline{\Omega}$, i.e., the exterior of the solution domain.

The above integral-representation formula is obtained from the Green's second identity using the fundamental solution $\Phi(\vec{x}, \vec{x}_o)$ of the adjoint equation to (1), satisfying

$$\mathcal{M}\left[\Phi(\vec{x}, \vec{x}_o)\right] = \nabla^2 \left[\Phi\right] - \frac{\partial(a\Phi)}{\partial x} - \frac{\partial(b\Phi)}{\partial y} + \mathrm{d}\Phi = -\frac{2\pi}{D}\delta(\vec{x}, \vec{x}_o) \qquad \forall \vec{x}, \vec{x}_o \in \Omega, \tag{3}$$

where δ is the Dirac delta function with pole at $\vec{x}_o = (x_o, y_o)$.

Using the complex variables $z = x + \mathrm{i}y$ and $\xi = x - \mathrm{i}y$, it is possible to transform the elliptic equation (1) into a hyperbolic equation (see [4,5]);

$$\overline{\mathcal{L}}\left[u\right] = \frac{\partial^2 u}{\partial z \partial \xi} + \mathcal{V}_z(z, \xi)\frac{\partial u}{\partial z} + \mathcal{V}_\xi(z, \xi)\frac{\partial u}{\partial \xi} + \mathcal{C}(z, \xi)u = \mathcal{F}(z, \xi), \tag{4}$$

where V_z, V_ξ, C and F are defined as

$$\begin{aligned} \mathcal{V}_z(z, \xi) &= \frac{1}{4}\left[a\left(\frac{z+\xi}{2}, \frac{z-\xi}{2i}\right) + \mathrm{i}b\left(\frac{z+\xi}{2}, \frac{z-\xi}{2i}\right)\right], \\ \mathcal{V}_\xi(z, \xi) &= \frac{1}{4}\left[a\left(\frac{z+\xi}{2}, \frac{z-\xi}{2i}\right) - \mathrm{i}b\left(\frac{z+\xi}{2}, \frac{z-\xi}{2i}\right)\right], \\ \mathcal{C}(z, \xi) &= \frac{1}{4}\left(d\frac{z+\xi}{2}, \frac{z-\xi}{2\mathrm{i}}\right), \quad \mathcal{F}(z, \xi) = \frac{1}{4}\, f\left(\frac{z+\xi}{2}, \frac{z-\xi}{2\mathrm{i}}\right). \end{aligned} \tag{5}$$

The complex form of the adjoint equation to (1) can be written as

$$\overline{\mathcal{M}}\left[v\right] = \frac{\partial^2 v}{\partial z \partial \xi} - \frac{\partial(\mathcal{V}_z(z, \xi)v)}{\partial z} - \frac{\partial(\mathcal{V}_\xi(z, \xi)v)}{\partial \xi} + \mathcal{C}v(z, \xi) = \mathcal{F}^*(z, \xi). \tag{6}$$

In terms of the complex-variable formulation, the fundamental solutions $S_u(x, y; x_o, y_o)$ and $\Phi(z, \xi; z_o, \xi_o)$ are particular solutions of the inhomogeneous equations

$$\overline{\mathcal{L}}[\mathcal{S}_u] = \frac{\partial^2 \mathcal{S}_u}{\partial z \partial \xi} + \mathcal{V}_z(z, \xi)\frac{\partial \mathcal{S}_u}{\partial z} + \mathcal{V}_\xi(z, \xi)\frac{\partial \mathcal{S}_u}{\partial \xi} + \mathcal{C}(z, \xi)\mathcal{S}_u = -\frac{\pi}{2D}\widetilde{\delta}(z - z_o)(\xi - \xi_o), \tag{7}$$

and

$$\overline{\mathcal{M}}[\Phi] = \frac{\partial^2 \Phi}{\partial z \partial \xi} - \frac{\partial(\mathcal{V}_z \Phi)}{\partial z} - \frac{\partial(\mathcal{V}_\xi \Phi)}{\partial \xi} + \mathcal{C}\Phi = -\frac{\pi}{2D}\widetilde{\delta}(z - z_o)(\xi - \xi_o), \tag{8}$$

where $\widetilde{\delta}$ is the delta function defined on the complex plane

The Riemann functions $R_u(z, \xi_o; z_o, \xi_o)$ and $R_v(z, \xi; z_o, \xi_o)$ are solutions of the homogeneous equations $\overline{\mathcal{L}}[\mathcal{R}_u] = 0$ and $\overline{\mathcal{M}}[\mathcal{R}_v] = 0$ satisfying given initial conditions. The representation of the fundamental solutions S_u and Φ in terms of the Riemann functions R_u and $R_v$presented in this section summarizes the works reported by Garabedian [5] and Vekua [4]. However, changes in the presentation and in some of the expressions are made in order to provide some additional results.

Using Leibnitz' rule, the homogeneous equation $\overline{\mathcal{M}}[\mathcal{R}_v] = 0$ can be rewritten as

$$\frac{\partial^2}{\partial z \partial \xi}\Bigg[\mathcal{R}_v(z, \xi; z_o, \xi_o) - \int_{\xi_o}^{\xi} \mathcal{V}_z(z, \eta)\mathcal{R}_v(z, \eta; z_o, \xi_o)\mathrm{d}\eta - \int_{z_o}^{z} \mathcal{V}_\xi(\gamma, \xi)\mathcal{R}_v(\gamma, \xi; z_o, \xi_o)\mathrm{d}\gamma$$
$$+ \int_{\xi_o}^{\xi} d\eta \int_{z_o}^{z} \mathcal{C}(\gamma, \eta)\mathcal{R}_v(\gamma, \eta; z_o, \xi_o)\mathrm{d}\gamma\Bigg] = 0. \tag{9}$$

The Riemann function $R_v$is a solution of this homogeneous equation satisfying the initial conditions

$$\frac{\partial \mathcal{R}_v(z, \xi_o; z_o, \xi_o)}{\partial z} - \mathcal{V}_\xi(z, \xi_o)\mathcal{R}_v(z, \xi_o; z_o, \xi_o) = 0,$$
$$\frac{\partial \mathcal{R}_v(z_o, \xi; z_o, \xi_o)}{\partial \xi} - \mathcal{V}_z(z_o, \xi)\mathcal{R}_v(z_o, \xi; z_o, \xi_o) = 0, \quad \mathcal{R}_v(z_o, \xi_o; z_o, \xi_o) = 1. \tag{10}$$

By direct integration of (9), taking into account the above initial conditions, we find that the Riemann function R_v is defined by the unknown density of a Volterra integral equation of the second kind,

$$\mathcal{R}_v(z, \xi; z_o, \xi_o) - \int_{\xi_o}^{\xi} \mathcal{V}_z(z, \eta)\mathcal{R}_v(z, \eta; z_o, \xi_o)\mathrm{d}\eta - \int_{z_o}^{z} \mathcal{V}_\xi(\gamma, \xi)\mathcal{R}_v(\gamma, \xi; z_o, \xi_o)\mathrm{d}\gamma$$
$$+ \int_{\xi_o}^{\xi} \mathrm{d}\eta \int_{z_o}^{z} \mathcal{C}(\gamma, \eta)\mathcal{R}_v(\gamma, \eta; z_o, \xi_o)\mathrm{d}\gamma = 1, \tag{11}$$

whose solution is given by a convergent Neumann series (for more details see [4]). As in the case of the fundamental solutions, closed-form expressions for the Riemann function R_v are known only for simple forms of the kernels $V_z(z, \eta)$, $V_\xi(\gamma, \xi)$ and $C(\gamma, \eta)$ in Eq. (11). Vekua [4] reported the closed-form expression for some special cases.

It can be shown by direct substitution that the following relation holds for any analytical function $u(z, \xi)$ that is a solution of (4) and the Riemann function R_v,

$$\frac{\partial^2 u \mathcal{R}_v}{\partial z \partial \xi} - R_v \overline{\mathcal{L}}[u] = \frac{\partial}{\partial z}\left[\frac{\partial \mathcal{R}_v}{\partial \xi}u - \mathcal{V}_z(z, \xi)\mathcal{R}_v u\right] + \frac{\partial}{\partial \xi}\left[\frac{\partial \mathcal{R}_v}{\partial z}u - \mathcal{V}_\xi(z, \xi)\mathcal{R}_v u\right]. \tag{12}$$

Integrating (12) by parts and using the initial conditions of the Riemann function R_v, i.e., Eq. (10), we can express the solution u of (4) in terms of the following Volterra integral representational formula,

$$\begin{aligned} u(z,\xi) &= u(z_o,\xi_o)\,\mathcal{R}_v(z_o,\xi_o;z,\xi) + \int_{z_o}^{z}\left[\mathcal{V}_\xi(\gamma,\xi_o)u(\gamma,\xi_o) + \frac{\partial u(\gamma,\xi_o)}{\partial\gamma}\right]\mathcal{R}_v(\gamma,\xi_o;z,\xi)\mathrm{d}\gamma \\ &\quad + \int_{\xi_o}^{\xi}\left[\mathcal{V}_z(z_o,\eta)u(z_o,\eta) + \frac{\partial u(z_o,\eta)}{\partial\eta}\right]\mathcal{R}_v(z_o,\eta;z,\xi)\mathrm{d}\eta \\ &\quad + \int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z}\mathcal{R}_v(\gamma,\eta;z,\xi)\,\overline{\mathcal{L}}\left[u(\gamma,\eta)\right]\mathrm{d}\gamma. \end{aligned} \tag{13}$$

This representation formula has been used previously by Young et al. [10] to solve boundary-value problems of Eq. (1), instead of using the classical formulation in terms of the Fredholm integral equation obtained from (2). The cases considered by Young et al. [10] correspond to those reported by Vekua [4] with the Riemann function R_v given in closed-form. Besides obtaining the Riemann function solution of the Volterra integral equation (11), the main difficulty of the Young et al. [10] approach lies in the implementation of the natural boundary conditions of the problem in (13).

Following the previous ideas, we find that the Riemann function R_u solution of $\overline{\mathcal{L}}\,[\mathcal{R}_u]=0$, satisfying the initial conditions

$$\begin{aligned} &\frac{\partial\mathcal{R}_u(z,\xi_o;z_o,\xi_o)}{\partial z} + \mathcal{V}_\xi(z,\xi_o)\mathcal{R}_u(z,\xi_o;z_o,\xi_o) = 0, \quad \frac{\partial\mathcal{R}_u(z_o,\xi;z_o,\xi_o)}{\partial\xi} + \mathcal{V}_z(z_o,\xi)\mathcal{R}_u(z_o,\xi;z_o,\xi_o) = 0, \\ &\mathcal{R}_u(z_o,\xi_o;z_o,\xi_o) = 1, \end{aligned} \tag{14}$$

is given by the unknown density of the following Volterra integral equation

$$\begin{aligned} &\mathcal{R}_u(z,\xi;z_o,\xi_o) + \int_{\xi_o}^{\xi}\mathcal{V}_z(z,\eta)\mathcal{R}_v(z,\eta;z_o,\xi_o)\mathrm{d}\eta + \int_{z_o}^{z}\mathcal{V}_\xi(\gamma,\xi)\mathcal{R}_v(\gamma,\xi;z_o,\xi_o)\mathrm{d}\gamma \\ &+ \int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z}\left(\mathcal{C}(\gamma,\eta) - \frac{\partial\mathcal{V}_z(\gamma,\eta)}{\partial\gamma} - \frac{\partial\mathcal{V}_\xi(\gamma,\eta)}{\partial\eta}\right)\mathcal{R}_u(\gamma,\eta;z_o,\xi_o)\mathrm{d}\gamma = 1. \end{aligned} \tag{15}$$

In order to obtain the above integral representation, we have rewritten the operator $\overline{\mathcal{L}}\,[]$ as

$$\overline{\mathcal{L}}\,[] = \frac{\partial^2\,[]}{\partial z\partial\xi} + \frac{\partial\mathcal{V}_z(z,\xi)\,[]}{\partial z} + \frac{\partial\mathcal{V}_\xi(z,\xi)\,[]}{\partial\xi} + \left(\mathcal{C}(z,\xi) - \frac{\partial\mathcal{V}_z(z,\xi)}{\partial z} - \frac{\partial\mathcal{V}_\xi(z,\xi)}{\partial\xi}\right)[]\,. \tag{16}$$

As in the case of (12), it can be shown that the following expression is satisfied by any analytical function $v(z,\xi)$ solution of (6) and the Riemann function R_u;

$$\frac{\partial^2 v\mathcal{R}_u}{\partial z\partial\xi} - R_u\overline{\mathcal{M}}\,[v] = \frac{\partial}{\partial z}\left[\frac{\partial\mathcal{R}_u}{\partial\xi}v + \mathcal{V}_z(z,\xi)\mathcal{R}_u v\right] + \frac{\partial}{\partial\xi}\left[\frac{\partial\mathcal{R}_u}{\partial z}v + \mathcal{V}_\xi(z,\xi)\mathcal{R}_u v\right]. \tag{17}$$

Integrating by parts and using the initial conditions of the Riemann function R_u, i.e., Eq. (14), we can express the solution v of (6) in terms of the Volterra integral-representational formula,

$$\begin{aligned} v(z,\xi) &= v(z_o,\xi_o)\,\mathcal{R}_u(z_o,\xi_o;z,\xi) + \int_{z_o}^{z}\left[\frac{\partial v(\gamma,\xi_o)}{\partial\gamma} - \mathcal{V}_\xi(\gamma,\xi_o)v(\gamma,\xi_o)\right]\mathcal{R}_u(\gamma,\xi_o;z,\xi)\mathrm{d}\gamma \\ &\quad + \int_{\xi_o}^{\xi}\left[\frac{\partial v(z_o,\eta)}{\partial\eta} - \mathcal{V}_z(z_o,\eta)v(z_o,\eta)\right]\mathcal{R}_u(z_o,\eta;z,\xi)\mathrm{d}\eta \\ &\quad + \int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z}\mathcal{R}_u(\gamma,\eta;z,\xi)\,\overline{\mathcal{M}}\left[v(\gamma,\eta)\right]\mathrm{d}\gamma. \end{aligned} \tag{18}$$

Garabedian [5] and Vekua [4] used the above Riemann functions and Volterra integral-representation formulae for the analytical functions u and v to obtain general expressions for the fundamental solutions $S_u(z,\xi;z_o,\xi_o)$ and $\Phi(z,\xi;z_o,\xi_o)$. In this way, the fundamental solution $S_u(z,\xi;z_o,\xi_o)$ is expressed as

$$\mathcal{S}_u(z,\xi;z_o,\xi_o)=\frac{1}{D}\mathcal{R}_u(z,\xi;z_o,\xi_o)\log\frac{1}{r}+\mathcal{B}_u(z,\xi;z_o,\xi_o), \tag{19}$$

where B_u is a regular function in some neighborhood of the source point (z_o,ξ_o). The above fundamental solution becomes logarithmically singular as $z\to z_o$ and $\xi\to\xi_o$, i.e., as $x\to x_o$ and $y\to y_o$, since $R_u(z_o,\xi_o;z_o,\xi_o)=1$. The limit $z\to z_o$implies $\xi\to\xi_o$, according to the definition of z and ξ.

Substituting expression (19) in (7) yields

$$\overline{\mathcal{L}}\,[\mathcal{B}_u]=\frac{1}{2}\left(\frac{\partial\mathcal{R}_u/\partial z+\mathcal{V}_\xi\mathcal{R}_u}{\xi-\xi_o}+\frac{\partial\mathcal{R}_u/\partial\xi+\mathcal{V}_z\mathcal{R}_u}{z-z_o}\right), \tag{20}$$

where we have noted that the Riemann function R_u is a solution of $\overline{\mathcal{L}}\,[\mathcal{R}_u]=0$, satisfying the initial condition (14), and $\log(1/r)$ is the fundamental solution of the Laplace equation in the complex plane,

$$4\frac{\partial^2\log(1/r)}{\partial z\partial\xi}=-2\pi\widetilde{\delta}(z-z_o)(\xi-\xi_o);\qquad\text{where } r^2=(z-z_o)(\xi-\xi_o), \tag{21}$$

The inhomogenous term in (20) is regular even in the limit $\xi\to\xi_o$ or $z\to z_o$, according to the initial conditions of the Riemann function R_u, i.e., Eq. (14). Therefore, a general solution of (20) can be given by the Volterra integral-representational formula (13) in terms of the Riemann function R_v. In particular, when the function $B_u(z,\xi;z_o,\xi_o)$ satisfies the initial condition

$$\frac{\partial\mathcal{B}_u(z_o,\xi;z_o,\xi_o)}{\partial\xi}+\mathcal{V}_z(z_o,\xi)\mathcal{B}_u(z_o,\xi;z_o,\xi_o)=0,\quad\frac{\partial\mathcal{B}_u(z,\xi_o;z_o,\xi_o)}{\partial z}+\mathcal{V}_\xi(z,\xi_o)\mathcal{B}_u(z,\xi_o;z_o,\xi_o)=0,$$
$$\mathcal{B}_u(z_o,\xi_o;z_o,\xi_o)=0, \tag{22}$$

Eq. (13) reduces to

$$\mathcal{B}_u(z,\xi;z_o,\xi_o)=\int_{\xi_o}^{\xi}\int_{z_o}^{z}\frac{1}{2}\left(\frac{\partial\mathcal{R}_u/\partial\sigma+\mathcal{V}_\xi\mathcal{R}_u}{\tau-\xi_o}+\frac{\partial\mathcal{R}_u/\partial\tau+\mathcal{V}_z\mathcal{R}_u}{\sigma-z_o}\right)\mathcal{R}_v(\sigma,\tau;z,\xi)\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{23}$$

It can be observed that the fundamental solution S_u is not uniquely determined since different forms of the function B_u are obtained from the integral-representational formula (13) with different initial conditions. This non-uniqueness arises because the fundamental solution is a particular solution of an inhomogeneous PDE. Adding any solution of the corresponding homogeneous equation we can construct a different form of the fundamental solution. We will consider this non-uniqueness property of the fundamental solution in more detail later. An alternative approach of finding the above fundamental solution in terms of a domain Fredholm integral equation instead a Volterra integral equation is given by Courant and Hilbert [1].

Following the previous approach, it is possible to find an expression for the fundamental solution $\Phi(z,\xi;z_o,\xi_o)$ in terms of the Riemann function R_v, that is, a solution of the homogeneous equation $\overline{\mathcal{M}}\,[\mathcal{R}_v]=0$, and a regular function B_v, as

$$\Phi(z,\xi;z_o,\xi_o)=\frac{1}{D}R_v(z,\xi;z_o,\xi_o)\log\frac{1}{r}+B_v(z,\xi;z_o,\xi_o). \tag{24}$$

Substituting the expression (24) in (8) results in the following inhomogeneous equation for the function B_v

$$\overline{\mathcal{M}}\,[\mathcal{B}_v]=\frac{1}{2}\left(\frac{\partial\mathcal{R}_v/\partial z-\mathcal{V}_\xi\mathcal{R}_v}{\xi-\xi_o}+\frac{\partial\mathcal{R}_v/\partial\xi-\mathcal{V}_z\mathcal{R}_v}{z-z_o}\right), \tag{25}$$

whose inhomogeneous term is regular even in the limit $\xi \to \xi_o$ or $z \to z_o$, in accord with the initial conditions of the Riemann function R_v, i.e., Eq. (10). Following the same ideas, we can give the solution of (25) satisfying the initial conditions

$$\frac{\partial \mathcal{B}_v(z,\xi_o;z_o,\xi_o)}{\partial z} - \mathcal{V}_\xi(z,\xi_o)\mathcal{B}_v(z,\xi_o;z_o,\xi_o) = 0, \quad \frac{\partial \mathcal{B}_v(z_o,\xi;z_o,\xi_o)}{\partial \xi} - \mathcal{V}_z(z_o,\xi)\mathcal{B}_v(z_o,\xi;z_o,\xi_o) = 0,$$
$$\mathcal{B}_v(z_o,\xi_o;z_o,\xi_o) = 0, \tag{26}$$

explicitly as (see Eq. (18))

$$\mathcal{B}_v(z,\xi;z_o,\xi_o) = \int_{\xi_o}^{\xi}\int_{z_o}^{z}\frac{1}{2}\left(\frac{\partial\mathcal{R}_v/\partial\sigma - \mathcal{V}_\xi\mathcal{R}_v}{\tau-\xi_o} + \frac{\partial\mathcal{R}_v/\partial\tau - \mathcal{V}_z\mathcal{R}_v}{\sigma - z_o}\right)\mathcal{R}_u(\sigma,\tau;z,\xi)\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{27}$$

Other forms of the Garabedian and Vekua expressions of the fundamental solutions $\Phi(z,\xi;z_o,\xi_o)$ and $S_u(z,\xi;z_o,\xi_o)$ reported in the literature can be directly found from the above equations.

Bergman [6] gives an explicit expression for the complex form of the fundamental solution $\Phi(z,\xi;z_o,\xi_o)$ in terms of an infinite series. Bergman's approach is based on the change of variable

$$\Phi(z,\xi) = \Gamma(z,\xi)V(z,\xi), \tag{28}$$

where

$$\Gamma(z,\xi) = \exp\left[\int_{\xi_o}^{\xi}\mathcal{V}_z(z,\eta)\mathrm{d}\eta\right]\exp\left[\int_{z_o}^{z}\mathcal{V}_\xi(\gamma,\xi_o)\mathrm{d}\gamma\right]. \tag{29}$$

Substituting the above change of variable in (8), we find that the function V satisfies the differential equation

$$\left(\frac{\partial^2 V}{\partial z\partial\xi} + \frac{\partial UV}{\partial\xi} + \left(F - \frac{\partial U}{\partial\xi}\right)V\right) = -\frac{\pi\widetilde{\delta}(z-z_o)(\xi-\xi_o)}{2D\Gamma(z,\xi)} = -\frac{\pi}{2D}\widetilde{\delta}(z-z_o)(\xi-\xi_o), \tag{30}$$

where

$$U(z,\xi) = \int_{\xi_o}^{\xi}\left(\frac{\partial\mathcal{V}_z(z,\eta)}{\partial z} - \frac{\partial\mathcal{V}_\xi(z,\eta)}{\partial\eta}\right)\mathrm{d}\eta \tag{31}$$

and

$$F(z,\xi) = \mathcal{C}(z,\xi) - \mathcal{V}_z(z,\xi)\mathcal{V}_\xi(z,\xi) - \frac{\partial\mathcal{V}_\xi(z,\xi)}{\partial\xi}. \tag{32}$$

Integrating Eq. (30), Bergman [6] found the following expression for the fundamental solution $\Phi(z,\xi;z_o,\xi_o)$

$$\Phi(z,\xi;z_o,\xi_o) = \frac{-1}{D}\left[\log(z-z_o) + \sum_{n=1}^{\infty}\frac{Q^{(n)}(z,\xi)}{2^{2n}\beta(n,n+1)}\int_{z_o}^{z}(z-t)^{n-1}\log(t-z_o)\mathrm{d}t\right], \tag{33}$$

where

$$Q^n(z,\xi;z_o,\xi_o) = \int_{\xi_o}^{\xi}P^{(2n)}(z,\tau)\mathrm{d}\tau \quad (n\geq 1), \quad P^{(2)}(z,\xi) = -2F(z,\xi), \tag{34}$$

$$-\frac{1}{2}(2n+1)P^{(2n+2)}(z,\xi) = \frac{\partial P^{(2n)}(z,\xi)}{\partial z} + U(z,\xi)P^{(2n)}(z,\xi) + F(z,\xi)\int_{\xi_o}^{\xi}P^{(2n)}(z,\tau)\mathrm{d}\tau, \tag{35}$$

and $\beta(n,m)$ is the beta function.

Hill and Porter [7] pointed out that, although the above expression for the fundamental solution Φ is given explicitly, it does not generally provide a convenient form for numerical evaluation, since the coefficients $U(z,\xi)$, $F(z,\xi)$ and $P^{(2n)}(z,\xi)$ must be evaluated pointwise in order to carry out the above integrations, and it is necessary to find the value of the derivatives $\partial P^{(2n)}(z,\xi)/\partial z$, $\partial V_z(z,\xi)/\partial z$ and $\partial V_\xi(z,\xi)/\partial\xi$ (see Eqs. (31), (34) and (35)). As a result, the recurrent determination of the coefficient Q^n to a specified level of accuracy proves troublesome, as does the truncation of the infinite series for points sufficiently remote from the source point.

To avoid the difficulty with the evaluation of Bergman's expression of the fundamental solution Φ, Hill and Porter [7] proposed solving (30) from the numerical solution of the following inhomogeneous Volterra integral equation;

$$V(z,\xi)+\int_{z_o}^{z}U(\gamma,\xi)V(\gamma,\xi)\mathrm{d}\gamma+\int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z}\left(F(\gamma,\eta)-\frac{\partial U(\gamma,\eta)}{\partial\eta}\right)V(\gamma,\eta)\mathrm{d}\gamma=f(z,\xi;z_o,\xi_o), \tag{36}$$

where the function $f(z,\xi;z_o,\xi_o)$ used by Hill and Porter, was chosen as

$$f(z,\xi;z_o,\xi_o)=\begin{cases}-\dfrac{1}{2D}\log(z-z_o)(\xi-\xi_o), & \text{if } U=0,\\ -\dfrac{1}{D}\log(z-z_o), & \text{if } U\neq 0.\end{cases} \tag{37}$$

Equation (36) is obtained from the differential equation (30) by direct integration as done previously to obtain the integral equation (15) (for more details see (38) and (39)). We will see latter that the choice of a different inhomogeneous term in (36) is justified by the non-uniqueness of the fundamental solution as a particular solution of (8). As can be observed, Eq. (36) is of the same form of the integral equations (11) and (15) for the Riemann functions R_v and R_u. Therefore the numerical scheme proposed by Hill and Porter [7] can be directly used to find the Riemann functions and the corresponding fundamental solutions given by (19) and (24).

Hill and Porter [7] used the proposed numerical solution of the above integral equation to find the fundamental solution of the Helmholtz equation with variable coefficients. Hill and Riley [9] and Hill et al. [8] used this numerical technique to solve problems of compressible potential flow. The numerical examples considered by Hill and coworkers correspond to cases defined by self-adjoint differential operators ([7]) or cases that can be converted into self-adjoint form ([8,9]).

Clements [3] obtained a closed-form expression of the fundamental solution Φ corresponding to a self-adjoint anisotropic diffusion equation with coefficients given by a quadratic polynomial. His approach can be regarded as a particular case of Bergman's approach [6] in the real plane, where the function Γ is given by the square root of the variable coefficient.

It is important to point out that, in most practical cases, the components $V_z(z,\xi)$ and $V_\xi(z,\xi)$ of the convection velocity are known as point values. Theses velocity values are generally obtained from other numerical solutions or found experimentally. In this respect, part of the difficulties encountered with the evaluation of the $Q^{(n)}$ coefficients on Bergman's series is also presented in Hill and co-workers' numerical approach when evaluating the functions $U(z,\xi)$ and $F(z,\xi)$, where it is necessary to find numerically the first derivative of the components of the convection velocity.

In the next section, we use some of the ideas of Garabedian and Vekua to obtain the Riemann functions R_u and R_v, and derive the fundamental solutions Φ and S_u as the unknown densities of uniquely solvable Volterra integral equations of the second kind whose analytical solutions are given explicitly by a convergent Neumann series. In our approach, it is not necessary to introduce an additional changes of variable, as in the Hill and co-workers' numerical approach. The obtained integral equation for the fundamental solution Φ depends on the local value of the components of the convection velocity and is independent of their first derivatives.

3 Fundamental solutions as Neumann series

In this section, we obtain the fundamental solutions Φ and S_u as the unknown density of two Volterra integral equations of the second kind obtained by directly integrating the differential equations (8) and (7) without any additional change of variable.

As in the case of Eq. (9), making use of Leibnitz' rule, we rewrite Eq. (8) as

$$\frac{\partial^2}{\partial z \partial \xi}\left[\Phi(z,\xi) - \int_{\xi_o}^{\xi} \mathcal{V}_z(z,\eta)\Phi(z,\eta)\mathrm{d}\eta - \int_{z_o}^{z} \mathcal{V}_\xi(\gamma,\xi)\Phi(\gamma,\xi)\mathrm{d}\gamma + \int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z} \mathcal{C}(\gamma,\eta)\Phi(\gamma,\eta)\mathrm{d}\gamma\right] = -\frac{\pi}{2D}\widetilde{\delta}(z-z_o)(\xi-\xi_o). \tag{38}$$

Taking into account that $\log(1/r)$ is the fundamental solution of the Laplace equation (see Eq. (21)), we can directly integrate the above equation to obtain a Volterra integral-representational formula for the fundamental solution,

$$\Phi(z,\xi) - \int_{\xi_o}^{\xi} \mathcal{V}_z(z,\eta)\Phi(z,\eta)\mathrm{d}\eta - \int_{z_o}^{z} \mathcal{V}_\xi(\gamma,\xi)\Phi(\gamma,\xi)\mathrm{d}\gamma + \int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z} \mathcal{C}(\gamma,\eta)\Phi(\gamma,\eta)\mathrm{d}\gamma = -\frac{1}{2D}\log(z-z_o)(\xi-\xi_o). \tag{39}$$

The function $\Phi(z,\xi)$ satisfies the initial conditions

$$\frac{\partial\Phi(z,\xi_o)}{\partial z} - \mathcal{V}_\xi(z,\xi_o)\Phi(z,\xi_o) = \frac{-1}{2D(z-z_o)}, \qquad \frac{\partial\Phi(z_o,\xi)}{\partial \xi} - \mathcal{V}_z(z_o,\xi)\Phi(z_o,\xi) = \frac{-1}{2D(\xi-\xi_o)}. \tag{40}$$

Therefore, as expected:

$$\lim_{(z,\xi)\to(z_o,\xi_o)} \Phi(z,\xi) = -\frac{1}{2D}\log(z-z_o)(\xi-\xi_o). \tag{41}$$

In contrast with the approach of Hill et al., the above integral equation involves only the point value of the components of the convection velocity and the reaction coefficient. It is important to point out that (39) is an integral equation of the Volterra type with a continuous inhomogeneous term, except at the point $z = z_o$ and $\xi = \xi_o$. In the previous section, we ensured the existence and uniqueness of this type of equation when V_z, V_ξ, K, and the inhomogeneous term F are analytic functions. To remove the singularity of the inhomogeneous term in the integral equation, i.e., $\log(z-z_o)(\xi-\xi_o)$, we define a new unknown function, $\Phi^*(z,\xi)$,

$$\Phi(z,\xi) = \Phi^*(z,\xi) - \frac{1}{2D}\log(z-z_o)(\xi-\xi_o), \tag{42}$$

where Φ^* is a regular function in some neighborhood of the point (z_o,ξ_o). Substituting this expression in (39), we obtain a new integral equation with an analytical inhomogeneous term,

$$\Phi^*(z,\xi) - \int_{\xi_o}^{\xi} \mathcal{V}_z(z,\eta)\Phi^*(z,\eta)\mathrm{d}\eta - \int_{z_o}^{z} \mathcal{V}_\xi(\gamma,\xi)\Phi^*(\gamma,\xi)\mathrm{d}\gamma + \int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z} \mathcal{C}(\gamma,\eta)\Phi^*(\gamma,\eta)\mathrm{d}\gamma = -\frac{1}{2D}h(z,\xi), \tag{43}$$

where

$$h(z,\xi) = \int_{\xi_o}^{\xi} \mathcal{V}_z(z,\eta)\log(z-z_o)(\eta-\xi_o)\mathrm{d}\eta + \int_{z_o}^{z} \mathcal{V}_\xi(\gamma,\xi)\log(\gamma-z_o)(\xi-\xi_o)\mathrm{d}\gamma - \int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z} \mathcal{C}(\gamma,\eta)\log(\gamma-z_o)(\eta-\xi_o)\mathrm{d}\gamma. \tag{44}$$

The solution of this integral equation can be obtained by the method of successive approximations as a Neumann series, providing us with an explicit form of the fundamental solution

$$\Phi(z,\xi) = -\frac{1}{2D}\log(z-z_o)(\xi-\xi_o) + \lim_{n\to\infty}\Phi_n^*(z,\xi), \tag{45}$$

where $\Phi_0^*(z,\xi) = -\frac{1}{2D}h(z,\xi)$ and

$$\Phi_n^*(z,\xi) = -\frac{1}{2D}h(z,\xi) + \int_{\xi_o}^{\xi}\mathcal{V}_z(z,\eta)\Phi_{n-1}^*(z,\eta)\mathrm{d}\eta + \int_{z_o}^{z}\mathcal{V}_\xi(\gamma,\xi)\Phi_{n-1}^*(\gamma,\xi)\mathrm{d}\gamma$$
$$-\int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z}\mathcal{C}(\gamma,\eta)\Phi_{n-1}^*(\gamma,\eta)\mathrm{d}\gamma, \tag{46}$$

for $n \geq 1$. The proof of the absolute and uniform convergence of the above series is given by Vekua [4], showing that the $\lim_{n\to\infty}\Phi_n^*(z,\xi)$ of the successive approximations exists and satisfies the integral equation (43). In contrast with Bergman's [6] infinite series and the Hill and Porter [7] numerical solution, the above Neumann series does not require the evaluation of the first derivative of the integral kernels for the evaluation of the recursive terms.

As a validation of the proposed approach, in the next section we consider a constant velocity field. For more general convection velocity fields, we recommend a numerical evaluation of the corresponding Neumann series using any integration approach available in the literature (see [11]), or direct numerical solution of the integral equation (39) using a numerical scheme similar to that proposed by Hill and Porter [7] for solving (36). Due to the non-uniqueness of the fundamental solution as a particular solution, it is possible to add to the resulting solution of the Volterra equation (39) a homogeneous solution of (6) to obtain a different fundamental solution.

Following the same approach, and using the operator $\overline{\mathcal{L}}[]$ in the form defined by (16), we can express the fundamental solution S_u of (7) in term of the following Volterra integral equation

$$\mathcal{S}_u(z,\xi;z_o,\xi_o) + \int_{\xi_o}^{\xi}\mathcal{V}_z(z,\eta)\mathcal{S}_v(z,\eta;z_o,\xi_o)\mathrm{d}\eta + \int_{z_o}^{z}\mathcal{V}_\xi(\gamma,\xi)\mathcal{S}_v(\gamma,\xi;z_o,\xi_o)\mathrm{d}\gamma$$
$$+\int_{\xi_o}^{\xi}\mathrm{d}\eta\int_{z_o}^{z}\left(\mathcal{C}(\gamma,\eta) - \frac{\partial\mathcal{V}_z(\gamma,\eta)}{\partial\gamma} - \frac{\partial\mathcal{V}_\xi(\gamma,\eta)}{\partial\eta}\right)\mathcal{S}_u(\gamma,\eta;z_o,\xi_o)\mathrm{d}\gamma = -\frac{1}{2D}\log(z-z_o)(\xi-\xi_o). \tag{47}$$

The solution is given by a Neumann series similar to that describing the solution of the integral equation (39).

3.1 Constant velocity

In the case of a convection velocity $\vec{v} = (v_x, 0)$ and zero reaction coefficient, $4V_z = 4V_\xi = -v_x/D$ and $C = 0$, the integral equation (39) reduces to

$$\Phi(z,\xi) + \frac{v_x}{4D}\int_{\xi_o}^{\xi}\Phi(z,\eta)\mathrm{d}\eta + \frac{v_x}{4D}\int_{z_o}^{z}\Phi(\gamma,\xi)\mathrm{d}\gamma = -\frac{1}{2D}\log\left[(z-z_o)(\xi-\xi_o)\right], \tag{48}$$

and the inhomogeneous term in (43) reduces to

$$-\frac{1}{2D}h(z,\xi) = -\frac{v_x}{4D^2}\left[\int_{\xi_o}^{\xi}\log(z-z_o)(\eta-\xi_o)\mathrm{d}\eta + \int_{z_o}^{z}\log(\gamma-z_o)(\xi-\xi_o)\mathrm{d}\gamma\right]. \tag{49}$$

By direct substitution, we show that the recursive term of the series, i.e., Eq. (46), has the explicit form

$$\Phi_n^*(z,\xi) = \frac{-1}{2D}\sum_{k=1}^{n}\frac{(-v_x/D)^k}{(k+1)!}\left[\mathfrak{Re}\,(z-z_o)\right]^k\left[\log\left[(z-z_o)(\xi-\xi_o)\right] - \sum_{j=1}^{k}\frac{2}{j+1}\right]. \tag{50}$$

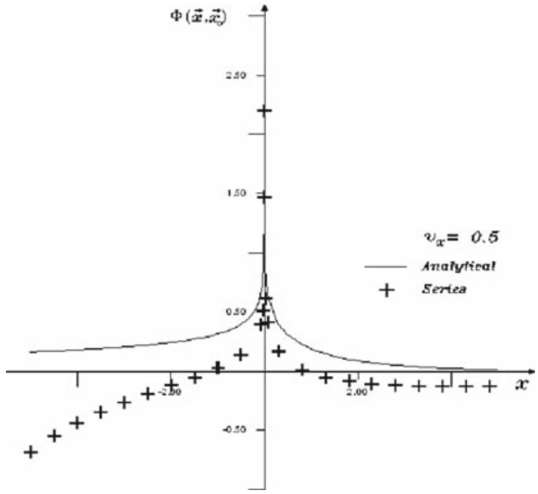

Fig. 1 Comparison of the closed-form analytical fundamental solution, Φ, and the Neumann series solution for $v_x = 0.5$

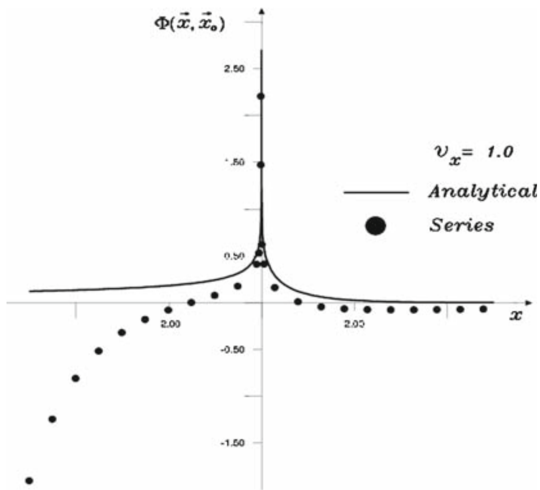

Fig. 2 Comparison of the closed-form analytical fundamental solution, Φ, and the Neumann series solution for $v_x = 1$

In this way, we derive the following expression for the fundamental solution in terms of real variables,

$$\Phi\left(\vec{x}, \vec{x}_o\right) = \frac{1}{D}\log\left(\frac{1}{r}\right) + \frac{1}{D}\sum_{k=1}^{\infty}\frac{(-v_x/D)^k}{(k+1)!}\left[(x-x_o)\right]^k\left[\log\left(\frac{1}{r}\right) + \sum_{j=1}^{k}\frac{1}{j+1}\right]. \tag{51}$$

Our approach is based on the numerical evaluation of a truncated form of the above infinite series. We compare the numerical solution obtained with the present approach to the well-known closed-form analytical solution (see [12])

$$\Phi(\vec{x}, \vec{x}_o) = \frac{1}{D}\exp\left(\frac{v_x(x-x_o)}{2D}\right)\mathcal{K}_o\left[\frac{|v_x|r}{2D}\right], \tag{52}$$

where the function K_o is the Bessel function of the second kind of order zero.

In our numerical examples, it was observed that the series converges fast when the evaluation point $\vec{x} = (x, y)$ is near the source point $\vec{x}_o = (x_o, y_o)$; only a few terms in the series are required. Additional terms are required for points far from the source point, or when the magnitude of the convection velocity is large. Figures 1 and 2 compare the closed-form expression of the fundamental solution to our numerical evaluation of the convergent series (51) for $v_x = 1$, $v_x = 0.5$, and $D = 1$. The source point is located at $\vec{x}_o = (0, 0)$, and the function is evaluated at points $(x, 0)$, in the interval $-5 \leq x \leq 5$.

The behavior of the obtained Neumann series is different from the known closed-form solution, except at the singular point. It is important to recall that the fundamental solution is not unique, even though the solution of the integral equation (39) is unique, as it is always possible to add to it a solution of the homogeneous form of (6).

A separation-of-variables solution of the homogeneous form of the adjoint equation to (1), with $a = -v_x/D$, $b = 0$, $d = 0$ and $f = 0$, is given by

$$v(x, y) = \left(C_1 + C_2\exp\left[-\frac{v_x}{D}(x-x_o)\right]\right)\left(C_3 + C_4(y-y_o)\right), \tag{53}$$

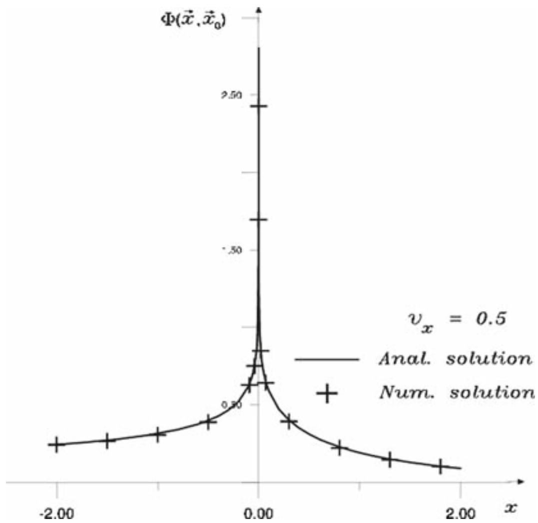

Fig. 3 Comparison of the closed-form analytical fundamental solution, Φ, and the modified Neumann series solution for $v_x = 0.5$

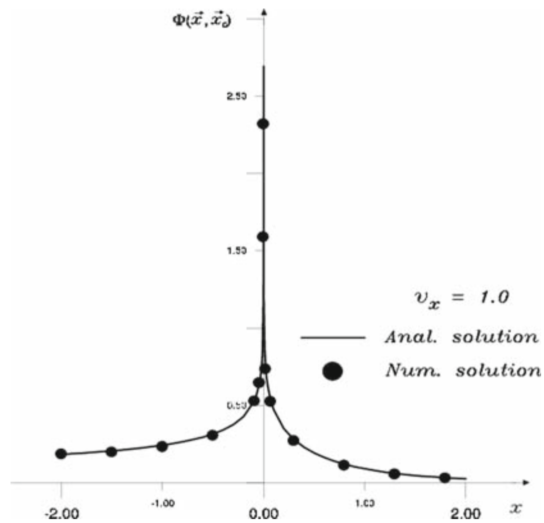

Fig. 4 Comparison of the closed-form analytical fundamental solution, Φ, and the modified Neumann series solution for $v_x = 1$

for arbitrary values of the constants C_i, with $i = 1, 2, 3, 4$. Due to the linearity of the problem, we can add to the previous Neumann series, i.e., Eq. (51), the above homogeneous solution and the new combination will still be a solution of (8). In this way, for the case under consideration, we find that the fundamental solution can be expressed as:

$$\Phi_n(\vec{x}, \vec{x}_o) = \log\left(\frac{1}{r}\right) + \sum_{k=1}^{\infty} \frac{(-v_x)^k}{(k+1)!} [(x - x_o)]^k \left[\log\left(\frac{1}{r}\right) + \sum_{j=1}^{k} \frac{1}{j+1}\right] + (C_1 + C_2 \exp[-v_x(x - x_o)])(C_3 + C_4(y - y_o)). \tag{54}$$

By choosing an appropriate set of constants, C_1, C_2, C_3 and C_4, it is possible to reproduce the behavior of the known closed-form expression of the fundamental solution given by (52). Figures 3 and 4 show results along the line $-2 \leq x \leq 2$; $y = 0$ for $v_x = 1$ and 0.5, with values of $C_1 = 0.095224$, $C_2 = 0.0232414$, $C_3 = 1$ and $C_4 = 0$ for $v_x = 0.5$ and $C_1 = 0.1544659$, $C_2 = 0.0759682$, $C_3 = 1$ and $C_4 = 0$ for $v_x = 1$.

In the present case of a constant convection velocity field, it is possible to find the solution of the Volterra integral equation (48) as a power series. This method entails of representing the solution of the integral equation (48) as

$$\Phi(z, \xi) = \sum_{n\geq 0}\sum_{m\geq 0} a_{nm}(z - z_o)^n(\xi - \xi_o)^m \log[(z - z_o)(\xi - \xi_o)] + \sum_{n\geq 0}\sum_{m\geq 0} b_{nm}(z - z_o)^n(\xi - \xi_o)^m, \tag{55}$$

where a_{nm} and b_{nm} are unknown constants to be determined. Substituting (55) in (48), we obtain the following expression for the coefficients a_{nm} and b_{nm},

$$a_{oo} = -\frac{1}{2D}, \quad a_{no} = -\frac{v_x}{4D}\frac{a_{(n-1)o}}{n}, n\geq 1, \quad a_{om} = -\frac{v_x}{4D}\frac{a_{o(m-1)}}{m}, \quad m\geq 1,$$

$$a_{nm} = -\frac{v_x}{4D}\left[\frac{a_{n(m-1)}}{m} + \frac{a_{(n-1)m}}{n}\right], \quad b_{nm} = -\frac{v_x}{4D}\left[\frac{b_{n(m-1)}}{m} + \frac{b_{(n-1)m}}{n} - \frac{a_{(n-1)m}}{n^2}\right], \quad n\geq 1, \quad m\geq 1,$$

$$b_{oo} = C, \quad b_{no} = -\frac{v_x}{4D}\left[\frac{b_{(n-1)o}}{n} - \frac{a_{(n-1)o}}{n^2}\right], \quad n\geq 1, \quad b_{om} = -\frac{v_x}{4D}\left[\frac{b_{o(m-1)}}{m} - \frac{a_{o(m-1)}}{m^2}\right], \quad m\geq 1,$$

where b_{oo} is a free term.

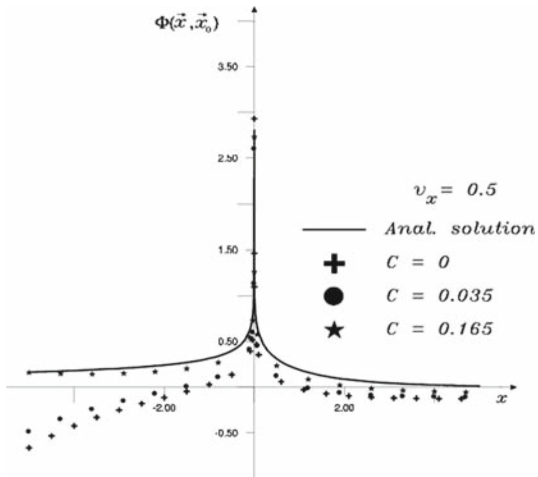

Fig. 5 Comparison between the analytical and power series solutions for $v_x = 0.5$ (source point at $\vec{x}_o = (0, 0)$ and evaluation point located at $\vec{x} = (x, 0)$, $-5 \le x \le 5$)

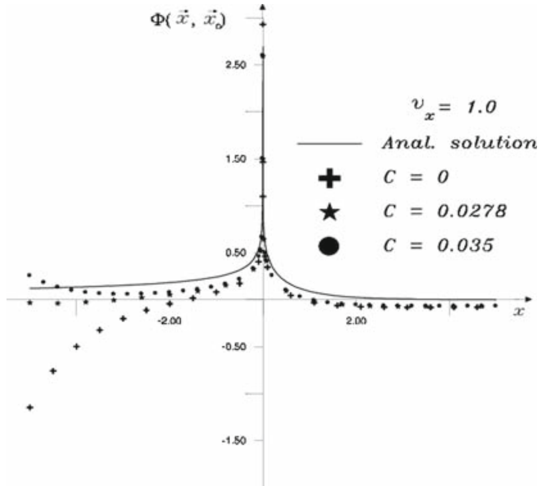

Fig. 6 Comparison between the analytical and power series solutions for $v_x = 1$ (source point at $\vec{x}_o = (0, 0)$ and evaluation point located at $\vec{x} = (x, 0)$, $-5 \le x \le 5$)

A comparison between the predicted values of Φ obtained by the power series and the closed-form analytical solution for the previous examples is made in Figs. 5 and 6 for different values of b_{oo}. The behavior of the power series is similar to that of the Neumann series when $C = 0$, as expected from the uniqueness of the solution of the integral equation. On the other hand, by choosing different value of C, we can modify the profile of the power series solution.

4 Conclusion

We have provided a review of classical methods for obtaining the Riemann and fundamental solutions of a two-dimensional convection–diffusion equation with variable coefficients and its adjoint equation, and proposed a new constructive way of obtaining the fundamental solutions of these two equations in terms of the unknown densities of two equivalent Volterra integral equations whose analytical solutions are given by convergent Neumann series.

As in the classical works of Vekua [4] and Garabedian [5] devoted to finding the Riemann functions $\mathcal{R}_u$ and $\mathcal{R}_v$, in our approach the fundamental solutions are defined in the complex plane in terms of a set of conjugate variables. The corresponding Volterra integral equations defining the fundamental solutions S_u and Φ are directly obtained from the complex form of the original differential equations, circumventing an additional change of variable used in Refs. [6,7].

The analytical solutions of the obtained integral equations are given by convergent Neumann series that define explicit expressions for the fundamental solutions, as in the case of Bergman's series [6], but without the difficulties encountered on the evaluation of the recursive terms of Bergman's series.

As a validation, the proposed approach was used to find the fundamental solution of the adjoint equation to the convection–diffusion equation with a constant velocity field. In this case, the resulting terms of the series can be evaluated analytically. For more involved variations of the velocity field, the recursive terms of the series can be evaluated by symbolic computation or numerical integration. Alternatively, the obtained Volterra integral equation can be solved numerically using a scheme similar to that proposed by Hill and Porter [7].

Since the fundamental solutions are particular solutions of the differential equations with delta function as inhomogeneous terms, it is possible to add to them a solution of the corresponding homogeneous equation. In our example, we added to the obtained Neumann series a solution of the homogeneous equation to obtain a new form of the fundamental solution. This new form of the fundamental solution shows excellent agreement with the known closed-form analytical solution for a constant velocity.

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